Math 251 **Methods of Integration**

This worksheet is to help students review the different methods of integration. It is assumed that students have seen all of these methods before.

The following is a list of integration formulas that you should know. Even when you go through the different methods, you ultimately reduce the integrals to one of these:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad n \neq -1$$

$$\int \frac{dx}{x} = \ln|x| + c$$

$$\int e^x dx = e^x + c$$

$$\int \sin x dx = -\cos x + c$$

$$\int \cos x dx = \sin x + c$$

$$\int \sec^2 x dx = \tan x + c$$

$$\int \csc^2 x dx = -\cot x + c$$

$$\int \sec x \tan x dx = \sec x + c$$

$$\int \csc x \cot x dx = -\csc x + c$$

The following are nice to know (make your life easier) integrals:

$$\int \tan x dx = -\ln|\cos x| + c$$

$$\int \cot x dx = \ln|\sin x| + c$$

$$\int \csc x dx = -\ln|\csc x + \cot x| + c$$

$$\int \sec x dx = \ln|\sec x + \tan x| + c$$

If a is constant, $a \neq 0$:

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + c \qquad \int \cos ax dx = \frac{1}{a} \sin ax + c \qquad \int \sin ax dx = -\frac{1}{a} \cos ax + c$$

Hint I: If you think you have an antiderivative, just differentiate and see if you end up with the integrand!

Hint II: The line of attack you should take when trying to integrate should be to try these methods in this order. You will always need to have paper and pencil, the correct method doesn't usually bounce off the page and hit you in the face!

Method 1. *Straight forward integration.* Using the following rules, the integrals can be reduced to one of the first set of integrals.

1. $\displaystyle\int cf(x)dx = c\int f(x)dx$

2. $\displaystyle\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

Example A. $\displaystyle\int (x^2+1)(x-3)dx = \int (x^3-3x^2+x-3)dx = \frac{x^4}{4} - x^3 + \frac{x^2}{2} - 3x + c$

Example B. $\displaystyle\int \frac{1-\sin^2 x}{1+\sin x}dx = \int \frac{(1-\sin x)(1+\sin x)}{1+\sin x}dx = \int (1-\sin x)dx = x + \cos x + c$

Problems:

1. $\displaystyle\int e^x(1-e^{-x}\sec^2 x)dx$

2. $\displaystyle\int \frac{x^3-2x+4}{x}dx$

3. $\displaystyle\int \cos x \tan x dx$

Method 2. *Substitution.* A simple substitution reduces the integral to one of the first set of integrals. In fact, the second set (the nice to know integrals) came about by using substitution. This method "undoes" the chain rule.

Example C. $\displaystyle\int \tan x dx = \int \frac{\sin x}{\cos x}dx$. Let $w = \cos x$ then $dw = -\sin x dx$ so that the integral becomes $\displaystyle\int -\frac{dw}{w} = -\ln|w| + c = -\ln|\cos x| + c$.

Example D. $\displaystyle\int xe^{x^2}dx$. Let $w = x^2$ then $dw = 2xdx$ so that the integral becomes $\displaystyle\int e^w\frac{dw}{2} = \frac{1}{2}e^w + c = \frac{1}{2}e^{x^2} + c$.

Hint III: Things to look for: if the integrand involves

$$e^{f(x)},\ \text{trig } (f(x)),\ \frac{1}{f(x)},\ (f(x))^n$$

Let $w = f(x)$. If the integrand involves $\ln x$ let $w = \ln x$. This is not an exclusive list. This is usually a good first try!

Hint IV: When substitution is complete, make sure no x's appear in the integrand.

Problems:

4. $\displaystyle\int \frac{\cos x}{1+\sin x}dx.$ Let $w = 1+\sin x$.

5. $\displaystyle\int (3x^2+x)\cos(2x^3+x^2+4)dx.$ Let $w = 2x^3+x^2+4$.

6. $\displaystyle\int \frac{(\ln x)^3}{x}dx.$ Let $w = \ln x$.

7. $\displaystyle\int \sec^2 x e^{\tan x} dx$

8. $\displaystyle\int x\sqrt{x+2}dx$

9. $\displaystyle\int \frac{x+1}{x^2+2x-5}dx$

Method 3. *Integration by parts.* This method "undoes" the product rule. The basic formula is

$$\int udv = uv - \int vdu$$

Example E. $\displaystyle\int x\ln x dx.$ Let $u = \ln x,\ dv = xdx$ then $du = \dfrac{dx}{x}$ and $v = \dfrac{x^2}{2}$.

$$\int x\ln x dx = (\ln x)\frac{x^2}{2} - \int \frac{x^2}{2}\frac{dx}{x} = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$$

Example F. $\displaystyle\int x\sin x dx.$ Let $u = x,\ dv = \sin x dx$. Then $dv = dx$ and $v = -\cos x$.

$$\int x\sin x dx = x(-\cos x) - \int -\cos x dx = -x\cos x + \sin x + c$$

Hint V: When choosing u and dv make sure dv is something that can be integrated. Also, the whole integrand should be taken up with u and dv (no extraneous x's around).

Hint VI: Method should be used when integrand involves

(poly)trig	$e^{ax}(\sin bx)$ $e^{ax}\cos bx$	$\sec^{(2n+1)} x$
(poly) $\ln x$	poly(inverse trig fact)	$\csc^{(2n+1)} x$

This is not an exclusive list!

Hint VII: The following may help you when trying to decide how to choose u and dv.

		u ↓	↑ dv
L	logarithms		
I	inverse trig		
A	algebraic		
T	trig		
E	exponential		

Look to see what expression appears first in the list above and that is u!

Problems:

10. $\int \ln x dx.$ Let $u = \ln x,\ dv = dx.$

11. $\int (x^2 + 2x - 1)\cos 3x dx.$ Let $u = x^2 + 2x - 1,\ dv = \cos 3x dx.$
    (Need to use integration by parts twice.)

12. $\int (x+3)e^{2x} dx.$ Let $u = x + 3,\ dv = e^{2x} dx.$

13. $\int x^2 \ln x dx$

14. $\int \tan^{-1} x dx$

15. $\int e^x \sin x dx$

Method 4. *Trigonometric integrals.* Integrands only involve trigonometric functions (not inverse trig functions!). Remember that certain trig functions "go together."

$$\sin\theta \text{ and } \cos\theta$$
$$\tan\theta \text{ and } \sec\theta$$
$$\cot\theta \text{ and } \cot\theta$$

If you have mixed trig functions convert everything to $\sin\theta$ and $\cos\theta$. Another helpful identity is $\sin^2\theta + \cos^2\theta = 1$. From here you can derive $\tan^2\theta + 1 = \sec^2\theta$ and $1 + \cot^2\theta = \csc^2\theta$. You should also have the half angle formulas in your repertoire.

$$\begin{aligned}\sin^2\theta =& \frac{1}{2}(1-\cos 2\theta)\\ \cos^2\theta =& \frac{1}{2}(1+\cos 2\theta)\end{aligned}$$

Most trigonometric integrations involve substitution. The only exceptions are when the integrand is $\sec^{2n+1}\theta$ or $\csc^{2n+1}\theta$ (see integration by parts) or $\sin^{2n}\theta\cos^{2m}\theta$ which involves the half angle formula (sometimes repeated several times).

Example G. $\int \sin\theta\cos^2\theta d\theta$. Let $w = \cos\theta; dw = -\sin\theta d\theta$

$$\int \sin\theta\cos^2\theta d\theta = \int -w^2 dw = -\frac{w^3}{3} + c = -\frac{\cos^3\theta}{3} + c$$

Example H. $\int \tan^2\theta d\theta = \int (\sec^2\theta - 1)d\theta = \tan\theta - \theta + c$

Example I. $\int \sin^2\theta\cos^3\theta d\theta$ Let $w = \sin\theta$ then $dw = \cos\theta d\theta$.

$$\int \sin^2\theta\cos^2\theta \underbrace{\cos\theta d\theta}_{dw} = \int w^2(1-\sin^2\theta)dw =$$

$$\int w^2(1-w^2)dw = \frac{w^3}{3} - \frac{w^5}{5} + c = \frac{\sin^3\theta}{3} - \frac{\sin^5\theta}{5} + c$$

Example J. $\int \sin^2\theta d\theta = \int \frac{1}{2}(1-\cos 2\theta)d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + c$

Hint VIII: As you can see there's not a whole lot of guessing to do with substitution. For instance in Example I above if you had let $w = \cos\theta$ then $dw = -\sin\theta d\theta$ and you have an "extra" $\sin\theta$ in the integrand. Of course it would be silly to guess $w = \tan\theta$ or $\sec\theta$ or another trig function. Use common sense and a paper and pencil!

Problems:

16. $\displaystyle\int \tan\theta\sec^3\theta d\theta$

17. $\displaystyle\int \cos^2\theta d\theta$

18. $\displaystyle\int \cot\theta\csc^3\theta d\theta$

19. $\displaystyle\int \sin\theta\cos\theta d\theta$

20. $\displaystyle\int \tan^3 2\theta\sec^2 2\theta d\theta$

21. $\displaystyle\int \sin^3\theta\cos^4\theta d\theta$

Note that integrands involving different arguments were not covered. i.e., $\displaystyle\int \cos 2x\sin 3x dx$. These type of integrals can be done using integration by parts twice and bringing the integral to the other side.

Method 5. *Trigonometric substitution.* You let $x =$ a trig function and change the integrand to one involving trig functions. The integrands involve no exponentials or trig functions to begin with. Rather they usually involve

$$(a^2 - x^2)^{n/2}$$
$$(x^2 - a^2)^{n/2}$$
$$(x^2 + a^2)^{n/2}$$

where a is constant and n is an integer.

If the integrand involves:	The substitution is:
$(a^2 - x^2)^{n/2}$	$x = a\sin\theta$
$(x^2 - a^2)^{n/2}$	$x = a\sec\theta$
$(a^2 + x^2)^{n/2}$	$x = a\tan\theta$

Example K. $\displaystyle\int \frac{dx}{4+x^2}$. Let $x = 2\tan\theta \quad dx = 2\sec^2\theta d\theta$.

$$\int \frac{dx}{4+x^2} = \int \frac{2\sec^2\theta d\theta}{4+4\tan^2\theta} = \int \frac{2\sec^2\theta d\theta}{4\sec^2\theta} = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + c = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

Example L. $\displaystyle\int (1-4x^2)^{1/2}dx \quad 2x = \sin\theta \quad 2dx = \cos\theta d\theta$

$$\int (1-4x^2)^{1/2}dx = \int (1-\sin^2\theta)^{1/2}\frac{\cos\theta d\theta}{2} = \frac{1}{2}\int \cos^2\theta d\theta = \frac{1}{2}\int \frac{1}{2}(1+\cos 2\theta)d\theta = \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta + c$$
$$\frac{1}{4}\theta + \frac{1}{4}\sin\theta\cos\theta + c = \frac{1}{4}\sin^{-1}(2x) + \frac{1}{4}(2x)(1-4x^2)^{1/2} + c$$

Hint IX: This type of substitution eliminates the + or - sign. Notice if you had guessed $x = 2\sin\theta$ in Example K, the denominator is $4 + 4\sin^2\theta$ which is not a trig identity and does not eliminate the + sign.

Problems:

22. $\displaystyle\int \frac{dx}{(x^2-1)^{1/2}}$

23. $\displaystyle\int \frac{dx}{x^2+2x+2}$

24. $\displaystyle\int \frac{e^x\,dx}{1+e^{2x}}$

25. $\displaystyle\int x^3(4+x^2)^{1/2}dx$

26. $\displaystyle\int \frac{dx}{\sqrt{2x-x^2}}$

Method 6. *Partial Fractions.* For this method the integrand can be written in the form $\dfrac{\text{polynomial}}{\text{polynomial}}$.

Steps:

1. Is the degree of the polynomial in the denominator greater than the degree of the polynomial in the numerator? If not, divide out.
2. Factor denominator.
3. Write the fraction as a sum of fractions.
4. Solve for constants.
5. Integrate.

Example M. $\displaystyle\int \frac{x-4}{x^2+5x+6}dx$

1. Yes.
2. $x^2+5x+6 = (x+2)(x+3)$
3. $\dfrac{x-4}{x^2+5x+6} = \dfrac{A}{x+2} + \dfrac{B}{x+3}$
4. $x-4 = A(x+3) + B(x+2)$

   Let $x=-3$ $\quad -3-4 = B(-3+2) \Rightarrow \quad B = 7$

   Let $x=-2$ $\quad -2-4 = A(-2+3) \Rightarrow \quad A = -6$
5. $\displaystyle\int \frac{(x-4)}{x^2+5x+6}dx = \int \left(\frac{-6}{x+2} + \frac{7}{x+3}\right) dx = -6\ln|x+2| + 7\ln|x+3| + c$

Example N. $\int \frac{(x^2+2x-4)}{x^3+x}dx$

Steps:

1. Yes.

2. $x^3+x = x(x^2+1)$

3. $\frac{x^2+2x-4}{(x+1)(x^2-x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

4. $x^2+2x-4 = A(x^2+1)+(Bx+C)x$

Set coefficients equal

$$\begin{array}{ll} \text{coef of } x^2 & 1 = A+B \\ \text{coef of } x & 2 = C \\ \text{const} & -4 = A \end{array}$$

Using $A=-4,\ C=2,\ B=5.$

5. $$\int \frac{x^2+2x-4}{x^3+x}dx = \int \frac{-4dx}{x} + \int \frac{5x+2}{x^2+1}dx = \int \frac{-4dx}{x} + \int \frac{5x}{x^2+1}dx + \int \frac{2}{x^2+1}dx$$
$$= -4\ln(x) + \frac{5}{2}\ln(x^2+1) = 2\arctan x + C$$

Problems:

27. $\int \frac{(x^2+4)dx}{x^3+3x^2+2x}$

28. $\int \frac{x+3}{x^3+x}dx$

29. $\int \frac{x-3}{x^3+4x^2+4x}dx$

30. $\int \frac{1}{x^3-x}dx$

Hint X: After factoring, the degree of the highest factor is usually not greater than 2.

Hint XI: Do not get caught up on the idea that the original integrand can be written as a $\dfrac{\text{polynomial}}{\text{polynomial}}$. Each of the following can be put into this form using substitution.

Problems:

31. $\displaystyle\int \frac{e^x\,dx}{e^{2x}+3e^x+2}$. Let $u = 3e^x$.

32. $\displaystyle\int \frac{\sin x dx}{\cos^3 x + \cos x}$

33. $\displaystyle\int \frac{dx}{x^{5/2}+7x^{3/2}+12x^{1/2}}$

The above list is all of the "typical" methods taught. After this, you should not be afraid to just take a pencil and paper and try different things. A Friday night problem $\displaystyle\int \sqrt{\tan x}dx$.