Math 251 Lines and Planes Name______________________

K. Pericak-Spector

The dot product and cross product are very useful tools when finding equations of lines and planes in 3-D. Let $\mathbf{U} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{V} = \langle v_1, v_2, v_3 \rangle$.

$$\mathbf{U} \cdot \mathbf{V} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$\mathbf{U} \cdot \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \mathbf{i}(u_2 v_3 - u_3 v_2) - \mathbf{j}(u_1 v_3 - v_1 u_3) + \mathbf{k}(u_1 v_2 - v_1 u_2)$$

Also recall that to find the vector between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, $\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

Problems:

1. Let $\mathbf{U} = \langle 1, 2, 3 \rangle$, $\mathbf{V} = \langle -1, 3, 4 \rangle$, $\mathbf{W} = \langle 2, -1, -1 \rangle$. Find:

a) $\mathbf{U} \cdot \mathbf{V}$ b) $\mathbf{U} \cdot \mathbf{W}$ c) $\mathbf{V} \cdot \mathbf{W}$

d) $\mathbf{U} \times \mathbf{V}$ e) $\mathbf{U} \times \mathbf{W}$ f) $\mathbf{V} \times \mathbf{W}$

2. Let $A(1, 3, -4)$, $B(2, -1, 3)$, and $C(4, 1, 1)$. Find

a) $\overrightarrow{AB}$ b) $\overrightarrow{AC}$ c) $\overrightarrow{BC}$ d) $\overrightarrow{AB} \cdot \overrightarrow{AC}$ e) $\overrightarrow{AB} \cdot \overrightarrow{BC}$

f) $\overrightarrow{AC} \cdot \overrightarrow{BC}$ g) $\overrightarrow{AB} \times \overrightarrow{AC}$ h) $\overrightarrow{AB} \times \overrightarrow{BC}$ i) $\overrightarrow{AC} \times \overrightarrow{BC}$

Answers:

1. a) 17 b) -3 c) $-g$ d) $\langle -1, 7, 5 \rangle$ e) $\langle 1, 7, -5 \rangle$ f) $\langle 1, 7, -5 \rangle$

2. a) $\langle 1, -4, 7 \rangle$ b) $\langle 3, -2, 5 \rangle$ c) $\langle 2, 2, -2 \rangle$ d) 46 e) -20 f) -8

g) $\langle -6, 16, 10 \rangle$ h) $\langle -6, 16, 10 \rangle$ i) $\langle -6, 16, 10 \rangle$

The equation of a line can be written as

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned} \quad \text{or} \quad \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

where (x_0, y_0, z_0) is a point on the line and $\langle a, b, c\rangle$ is the direction. To find the equation of a line you need two things–a point $P(x_0, y_0, z_0)$ and a direction of the line $\mathbf{U} = \langle a, b, c\rangle$. Then

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}$$

Examples:

A. Find the equation of the line which contains the points $A(1, -1, 4)$ and $B(-3, 2, 5)$. You can use either points as (x_0, y_0, z_0). To find the direction, find the vector from A to B $\overrightarrow{AB} = \langle -4, 3, 1\rangle$. Then

$$\begin{aligned} x &= 1 - 4t \\ y &= -1 + 3t \\ z &= 4 + t \end{aligned}$$

B. Find the equation of the line parallel to

$$\begin{aligned} x &= 4 + 2t \\ y &= 3 - t \\ z &= 2 + 7t \end{aligned}$$

through the point $(4, -2, 3)$. The direction is given by the coefficients of $t, \mathbf{U} = \langle 2, -1, 7\rangle$. Then

$$\begin{aligned} x &= 4 + 2t \\ y &= -2 - t \\ z &= 3 + 7t \end{aligned}$$

The equation of a plane can be written as

$$ax + by + cz = d$$

where $\langle a, b, c\rangle$ denote the direction perpendicular to plane. To find the equation of a plane you need two things: a point $P(x_0, y_0, z_0)$ and the perpendicular direction ($\mathbf{N}$). Then, if $Q(x, y, z)$ is <u>any</u> point in the plane $\overrightarrow{PQ} = \langle x - x_0, y - y_0, z - z_0\rangle$ is a vector in the plane and is perpendicular to $\mathbf{N}$. i.e., $\mathbf{N} \cdot \overrightarrow{PQ} = 0$.

C. Find the equation of the plane containing the points $A(1, -1, 4)$, $B(-3, 2, 5)$ and $C(1, 3, 8)$. If A, B and C are points in the plane $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{BC}$ are vectors in the plane $\overrightarrow{AB} = \langle -4, 3, 2\rangle$, $\overrightarrow{AC} = \langle 0, 4, 4\rangle$. $\overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{AC}$. i.e., it is perpendicular to the plane.

$$\mathbf{N} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 3 & 1 \\ 0 & 4 & 4 \end{vmatrix} = \mathbf{i}(8) - \mathbf{j}(-16) + \mathbf{k}(-16)$$

Let $Q(x,y,z)$ be any point in the plane. Then $\overrightarrow{AQ} = \langle x-1, y+1, z-4\rangle$ is a vector in the plane and $8(x-1)+16(y+1)-16(z-4)=0$.

D. Find the equation of the plane which contains the point $(1,3,6)$ and the line

$$\begin{aligned} x &= 1+t \\ y &= -2-t \\ z &= 4+3t \end{aligned}$$

The vector $\langle 1,-1,3\rangle = \mathbf{U}$ (the direction of the line) is in the plane. At $t=0$, $(1,-2,4)$ a point on the line and is in the plane. You now have two points in the plane. Let $A(1,3,6)$, $B(1,-2,4)$ and $\overrightarrow{AB} = \langle 0,-5,-2\rangle$ is in the plane

$$\mathbf{U}\times\overrightarrow{AB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ 0 & -5 & -2 \end{vmatrix} = \mathbf{i}(17) - \mathbf{j}(-2) + \mathbf{k}(-5) = \mathbf{N}$$

Let $Q(x,y,z)$ then $\overrightarrow{AQ} = \langle x-1, y-3, z-6\rangle$ and $\overrightarrow{AQ}\cdot\mathbf{N} = 17(x-1)+2(y-3)-5(z-6)=0$.

Now that you know the basics, see if you can do the following problems. (These came from old tests!)

1. Find the equation of the plane which contains the lines

$$\begin{aligned} x &= 1+t \\ y &= -2+2t \\ z &= 6-t \end{aligned} \qquad \text{and} \qquad \begin{aligned} x &= 2+s \\ y &= 3+2s \\ z &= -1-s \end{aligned}$$

2. Find the equation of the line parallel to the above lines which goes through the point (0,0,0).

3. Determine if the line and plane intersect and, if so, find the point of intersection.

$$\begin{aligned} x &= 3+2t \\ y &= 4-6t \\ z &= -7+4t \end{aligned} \qquad \text{and} \qquad x-3y+2z=61$$

   If they intersect, is the intersection orthogonal? Why or why not?

4. Find an equation of the plane perpendicular to the planes $x+y+z=7$ and $2x-3y+4z=8$ through the point $(-1,4,-3)$.

5. Find the equation of the line perpendicular to the lines at their point of intersection.

$$\begin{aligned} x &= 2+3t \\ y &= 6-t \\ z &= -1+2t \end{aligned} \qquad \begin{aligned} x &= 4+s \\ y &= -4+s \\ z &= 12-s \end{aligned}$$

6. a) Find the line of intersection of the planes

$$x + y + 2z = 3 \qquad x - y - z = 1$$

b) Find the point of intersection between this line and the plane

$$2x - y + 2z = -7.$$

c) What does this represent?

7. Find the equation of the plane which contains the point $(2, -1, -1)$ and the line

$x = 3 - t$

$y = 2 + t.$

$z = t$

8. Find an equation of the line that is perpendicular to the above plane and passes through the point $(-1, -2, -3)$.

9. Determine the point(s), if any, where the line and plane intersect.

$$\begin{aligned} x &= 1 + 3t \\ y &= 1 + t \\ z &= 3 - t \end{aligned} \qquad \text{and} \qquad 3x + y - z = 23$$

Is the intersection orthogonal?

10. Two of the lines intersect. Determine which two and find their point of intersection.

$$\begin{aligned} x &= 1 + 2t \\ y &= 3 - t \\ z &= 7 + 4t \end{aligned} \qquad \begin{aligned} x &= 3 + 2s \\ y &= 6 - s \\ z &= 5 + 4s \end{aligned} \qquad \begin{aligned} x &= 1 + 4p \\ y &= 1 \\ z &= 12 + 3p \end{aligned}$$

11. Find the equation of the plane through the point $(-1, 6, -5)$ and parallel to the plane $x+y+z+2 = 0$.

12. Find an equation of the line that is perpendicular to the above plane and passes through the point $(1, -2, 1)$.

13. Consider the two parallel lines

$$\begin{aligned} x &= 1 + 2t \\ y &= 1 - 2t \\ z &= 3 + t \end{aligned} \qquad \begin{aligned} x &= 2 + 2s \\ y &= -1 - 2s \\ z &= s \end{aligned}$$

You want to find the equation of the line perpendicular to these two lines through the point (1,1,3).

a) Find an equation of the plane which contains these two lines.

b) Take the cross product of the normal to this plane (from problem a) and the direction of the above lines.

c) From problem (b) you have the direction of the line perpendicular to both lines. Write the equation of the line.

d) Show that this new line hits both parallel lines.

e) Show that this new line is perpendicular to both lines.

14. Find the line of intersection of the planes $x + y + 2z = 3$ and $2x + y + z = 5$.

15. Find the equation of the plane that passes through the point $(1, 1, -1)$ and contains the line of intersection of the planes found above.

16. Determine if the following two lines intersect:

$$\begin{aligned} x &= 1 + t \\ y &= 2 - t \\ z &= 3t \end{aligned} \qquad\qquad \begin{aligned} x &= 2s \\ y &= -19 - 4s \\ z &= -7 - 2s \end{aligned}$$

If the lines intersect, do they intersect orthogonally?

17. Determine the point(s), if any, where the line and plane intersect.

$$\begin{aligned} x &= 2t \\ y &= 1 + 2t \\ z &= -4t \end{aligned} \qquad \text{and} \qquad x + 2y + z = 2$$

18. You are going to find the distance between the two planes

$$\begin{aligned} 4x + 2y + z &= 8 \\ 4x + 2y + z &= 29 \end{aligned}$$

in the following manner:

a) Find a point in the plane $4x + 2y + z = 8$.

b) Find an equation of a line through this point found in problem (a) and perpendicular to the planes.

c) Find the point where this line intersects the plane $4x + 2y + z = 29$.

d) Find the distance between the two points.

19. Find the equation of the plane through the point $(-1, 6, -5)$ and parallel to the plane $x+y+z+2 = 0$.

20. Find an equation of the line that is perpendicular to the above plane and passes through the point $(1, -2, 1)$.

21. Find the point of intersection of the two lines

$$\begin{aligned} x &= 3 + t \\ y &= 10 - 2t \\ z &= -14 + 2t \end{aligned} \qquad\qquad \begin{aligned} x &= s \\ y &= 4 - s \\ z &= 28 - 2s \end{aligned}$$

Do they intersect orthogonally?

22. Find the equation of the plane which contains these two lines from above.

23. Find the equation of the line which is perpendicular to the lines

$$\begin{aligned} x &= 4 - t \\ y &= 6 + 3t \\ z &= -1 - 2t \end{aligned} \qquad \begin{aligned} x &= 1 + 2t \\ y &= -1 + 4t \\ z &= -1 + 6t \end{aligned}$$

and goes through the point $(6, 1, -4)$.

24. We are going to find the distance from the point $(1, 1, 4)$ to the line

$$\begin{aligned} x &= 3 + t \\ y &= 4 - 2t \\ z &= 6 - t \end{aligned}$$

   a) Find the equation of the plane through $(1, 1, 4)$ and perpendicular to the line.

   b) Now that you have a plane and a line through the plane, find where the line and plane intersect.

   c) You now have the point on line which is closest to $(1, 1, 4)$. Find the distance between these two points.

25. Find the equation of the line through the point $(1, -1, 2)$ and parallel to the planes $x - y + z = 4$ and $x - y + z = 0$.

26. Find a plane through the points $P(1, 2, 3)$ and $Q(3, 2, 1)$ and perpendicular to the plane $4x - y + 2z = 7$.

27. Determine the point of intersection between the plane $2x + 4y - z = 7$ and the line

$$\begin{aligned} x &= 1 + t \\ y &= t \\ z &= 7 + 2t \end{aligned}$$

Does the line intersect the plane orthogonally? Why or why not?

28. Find a plane perpendicular to the planes $3x - 2y + 7z = 4$ and $x + y - 3z = 1$ and through the point $(1, -1, 4)$.

29. Find the equation of the line of intersection of the two planes $2x + y + 4z = 6$ and $x + y + z = 10$.

    Do the planes intersect orthogonally? Why or why not?

30. a) Determine where the lines intersect.

$$\begin{aligned} x &= -1 + t \\ y &= 3 - 2t \\ z &= -2 + 6t \end{aligned} \qquad \begin{aligned} x &= 7 + 5s \\ y &= -2 + s \\ z &= 14 - 2s \end{aligned}$$

   b) Find the equation of the plane containing these two lines.

Answers:

1. $-gx + 6y + 3z = -3$ 2. $x = t,\ y = 2t,\ z = -t$

3. The intersection point is $(g, -14, 5)$ Yes, they intersect orthogonally. Because V//N.

4. $7x - 2y - 5z = 0$ 5. $x = 11 - t,\ y = 3 + 5t,\ z = 5 + 4t$

6. a) $x = 2 + t,\ y = 1 + 3t,\ z = -2t$ b) $(4, 7, -4)$

7. $x - y + 2z = 1$ 8. $x = -1 + t,\ y = -2 - t,\ z = -3 + 2t$ 9. $(7, 3, 1)$

10. 1st and 3rd lines intersect at the point $(5, 1, 15)$

11. $x + y + z = 0$ 12. $x = 1 + t,\ y = -2 + t,\ z = 1 + t$

13. $x = 1 - 8t,\ y = 1 - 7t,\ z = 3 + 2t$

14. $x = 3 - t,\ y = 1 + 3t,\ z = -t$

15. $x - z = 2$ 16. They don't intersect.

17. $(2a, 1 + 2a, -4a)$ for all $a \in \mathbb{R}$

18. a) $(1, 2, 0)$ b) $x = 1 + 4t,\ y = 2 + 2t,\ z = t$ c) $(5, 4, 1)$ d) $\sqrt{21}$

19. $x + y + z = 0$ 20. $x = 1 + t,\ y = 2 + t,\ z = 1 + t$

21. $(12, -8, 4)$ They don't intersect orthogonally.

22. $6x + 4y + z = 44$ 23. $x = 6 + 26t,\ y = 1 + 2t,\ z = -4 - 10t$

24. a) $x - 2y - z = -5$ b) $(4, 2, 5)$ c) $\sqrt{11}$

25. $x = 1 + 2t,\ y = -1,\ z = 2 - 2t$ 26. $x + 6y + z = 16$ 27. $(4, 3, 13)$

28. $-x + 16y + 5z = 3$ 29. $x = -4 - 3t,\ y = 14 + 2t,\ z = t$

30. a) $(2, 0, 16)$ b) $-2x + 32y + 11z = 172$