Math 251 Derivatives Name______________________
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Let $f(x, y, z)$ be a scaler-valued function then ∇f is a vector where $\nabla f = \langle f_x, f_y, f_z \rangle$.
Examples:

1.

$$f(x, y, z) = x^2y + 3xyz^3 + 2y^2z$$
$$\nabla f = \langle 2xy + 3yz^3 m\ x^2 + 3xz^3 + 4yz,\ 9xyz^2 + 2y^2 \rangle$$

2.

$$f(x, y, z) = \frac{x^2y}{z^3}$$
$$\nabla f = \langle \frac{2xy}{z^3}, \frac{x^2}{z^3}, \frac{-3x^2y}{z^4} \rangle$$

Directional Derivatives You want to determine how quickly a function is increasing or decreasing in a particular direction.

The directional derivative of f in the direction $\mathbf{U}$ is given by

$$D_uf = \nabla f \cdot \mathbf{U} \text{ where } \mathbf{U} \text{ is a unit vector.}$$

Examples:

3. Find the directional derivative of $f(x, y) = x^2 + 2xy + y^3$ at the point $(-1, 2)$ in the direction $\mathbf{V} = \langle 1, 2 \rangle$.

Now $\nabla f = \langle 2x + 2y, 2x + 3y^2 \rangle$ at the point $(-1, 2)\nabla f(-1, 2) wehavethat = \langle 2, 10 \rangle$. We need to make $\mathbf{V}$ a unit vector so $|\mathbf{V}| = \sqrt{1+4} = \sqrt{5}$ and $\mathbf{U} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$.
Now $D_uf = \langle 2, 10 \rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \frac{2}{\sqrt{5}} + \frac{20}{\sqrt{5}} = \frac{22}{\sqrt{5}}$.

4. Find the directional derivative of $f(x, y, z) = x^3z - xy^2 + 4z^3$ at $A(-1, 1, 3)$ in the direction of the point $B(1, -1, 4)$.

Now $\nabla f = \langle 3x^2z - y^2, -2xy, x^3 + 12z^2 \rangle$ at the point $A, \nabla f(-1, 1, 3) = \langle 8, 2, 107 \rangle$. The direction of the vector is $\overrightarrow{AB} = \langle 2, -2, 1 \rangle$. We need to make it a unit vector so $|\overrightarrow{AB}| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$ so $\mathbf{U} = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$.

$$D_uf = \nabla f \cdot \mathbf{U} = \langle 8, 2, 107 \rangle \cdot \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle = \frac{16}{3} - \frac{4}{3} + \frac{107}{3} = \frac{119}{3}$$

Problems:

1. Find the directional derivative of $f(x,y) = xe^{xy}$ at the point $P(1{,}2)$ in the direction of the point $Q(-1,1)$.

2. Find the directional derivative of $f(x,y,z) = x^2 + y^2 - 4z$ at the point $P(1,2,-1)$ in the direction of the point $Q(4,3,5)$.

3. Find the directional derivative of $f(x,y,z) = 3x^2y - y^2z + z^3$ at the point $P(4,3,5)$ in the direction of the point $Q(1,2,-1)$.

4. a) Find the directional derivative of $f(x,y,z) = \ln(x^2 + y^3) + \sin \pi z$ at the point $P(1,2,-1)$ in the direction of the point $Q(4,4,5)$.

   b) What is the direction of maximum rate of increase?

5. a) Find the directional derivative of $f(x,y,z) = xy + y^2$ at the point $P(3,2)$ in the direction of the point $\mathbf{v} =< 3,4 >$

   b) In what direction is the directional derivative equal to zero?

6. a) Let $f(x,y,z) = xyz + 3x^2 - 2y^2 + z^2$. Find the directional derivative of f at the point $(1,-1,2)$ in the direction of the point (11, 10, 0).

   b) What is the **rate** of maximum increase at this point? (Not the direction!)

7. a) Let $f(x,y,z) = x^2yz + 3xy^2z^2$. Find the directional derivative of f at the point $(1,1,1)$ in the direction $\langle 3,4,5 \rangle$.

   b) Find the unit vector in the **direction** of maximum increase at this point.

8. a) Suppose the temperature in degrees Celsius on the surface of a metal plate is

$$T(x,y) = 20 - 4x^2 - 4y^2$$

   where x and y are measured in centimeters.

   1. In what direction from $(2,-3)$ does the temperature increase most rapidly?
      I.e., Find in which direction the directional derivative increases most rapidly.

   2. What is this rate of increase?

b) Now suppose a heat-seeking particle is located at the point $(2, -3)$ on a metal plate whose temperature at (x, y) is

$$T(x, y) = 20 - 4x^2 - 4y^2.$$

Find the <u>path</u> of the particle as it continuously moves in the direction of maximum temperature increase by using the following method: Let the path be represented by the position function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

1. What is the tangent vector for this $\mathbf{r}(t)$ at each $(x(t), y(t))$?

2. Find the ∇T.

3. Because the particle seeks maximum temperature increase, the directions of this tangent vector $\mathbf{r}'(t)$ and ∇T are the same at each point on the path. Set these vectors equal. I.e., Set $\mathbf{r}' = \nabla T$.

4. You should now have an equation for $\dfrac{dx}{dt}$ and one for $\dfrac{dy}{dt}$. Find
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

5. Solve for x and dx on one side and y and dy on the other.
I.e., Make sure dx and dy are in the numerators.

6. Integrate both sides.

7. Plug in $x = z,\ y = -3$ to solve for the constant of integration.

8. Now write as $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}$ by setting one of the variables equal to t.

<u>Tangent Plane and Normal Lines</u> To find the tangent plane to a curve we need a normal direction perpendicular to the curve and a point. To find a normal line we need a direction perepndicular to the curve and a point. The normal direction is given by ∇f.

Examples:

5. Find the tangent plane and normal line of

$$x^3y^2 - 3xz^2 = 2xyz = -3 \text{ at } A(1, 2, -1)$$

Let $f(x, y, z) = x^3y^2 - 3xz^2 + 2xyz + 3$ then

$$\nabla f = \langle 3x^2y^2 - 3z^2 + 2yz,\ 2x^3y + 2xz, -6xz + 2xy\rangle.$$

At the point A, $\nabla f(1,2,-1) = \langle 5,2,10\rangle = \mathbf{N}$. Let $Q(x,y,z)$ be any point in the plane. Then $\overrightarrow{AQ} = \langle x-1, y-2, z+1\rangle$ and the equation of the plane is given by

$$\mathbf{N}\cdot\overrightarrow{AQ} = 5(x-1) + 2(y-2) + 10(z+1) = 0$$

The equation of the normal line is given by

$$\begin{aligned} x &= 1+5t \\ y &= 2+2t \\ z &= -1+10t \end{aligned}$$

6. Find the equation of the tangent plane and normal line if

$$2xy + z^2 + \cos\pi x - 3y^2 = 3 \text{ at } A(2,1,-1)$$

Then let $f(x,y,z) = 2xy + z^2 + \cos\pi x - 3y^2 - 3$ then $\nabla f = \langle 2y - \pi\sin\pi x, 2x - 6y, 2z\rangle$. At the point A, $\nabla f(2,1,-1) = \langle 2,-2,-2\rangle = \mathbf{N}$. Let $Q(x,y,z)$ be any point in the plane then $\overrightarrow{AQ} = \langle x-2, y-1, z+1\rangle$ and the equation of the plane is given by

$$\mathbf{N}\cdot\overrightarrow{AQ} = 2(x-2) - 2(y-1) - 2(z+1) = 0$$

The equation of the normal line is given by

$$\begin{aligned} x &= 2+2t \\ y &= 1-2t \\ z &= -1-2t \end{aligned}$$

Problems:

9. a) Find the equation of the tangent plane to the surface at the point $(2,1,-2)$ to $xy^2+3x-z^2=4$.

   b) Find the equation of the normal line to the surface at this point.

10. Find the equation of the tangent plane to the surface $x^2+4y^2+z^2=36$ at the point $(2,-2,4)$.

11. The above surface is an ellipse. There are two tangent planes of this surface parallel to the $x-y$ plane. Find the equation of one of these planes.

12. a) Find the equation of the tangent plane to the surface at the point $(1,2,1)$ to $\tan^{-1} x+\ln(y^2+z)=\dfrac{\pi}{4}+\ln 5$.

    b) Find the equation of the normal line to the surface at this point.

13. a) Find the equation of the tangent plane to the surface at the point $\left(\dfrac{\pi}{2},-1,1\right)$ to $\tan^{-1} y+\sin xz^2-\dfrac{x}{2y}=1$.

    b) Find the line of intersection between the above plane and the plane $z=1$.

14. a) Find the equation of the tangent plane to the surface at the point $(0,\dfrac{1}{4},\dfrac{1}{6})$ to $e^{xz}+\tan^{-1}(2y+3z)=1+\dfrac{\pi}{4}$.

    b) Find the equation of the line in the $y-z$ plane.

    c) Find the equation of the normal line to the surface at this point.

15. Find the equation of the tangent plane to the surface at the point $(0,1,2)$ to $e^{xz}+\cos(\pi x)-x^2y+yz=4$.

16. The surface $x^2+y^2=4$ intersects the surface $x^2+y^2-z=0$ at $(\sqrt{2},\sqrt{2},4)$.

    a) Show $(\sqrt{2},\sqrt{2},4)$ is on both surfaces.

    b) Find the normal line to both surfaces at this point.

    c) Find the equation of the tangent line to the curve of intersection of these surfaces at this point. (Hint: You already have the point ant the normal to both surfaces. Remember the normal is perpendicular to the surface!)

17. a) Find the equation of the tangent plane to the surface at the point $(1,-1,4)$ to $xz^2+\ln(y^2+z^2)-\tan^{-1}\dfrac{x}{y}=4$.

    b) Find the unit vector perpendicular to the tangent plant to the above surface at this point.

    c) The tangent plane intersects the yz plane to form a line. Find the equation of this line.

18. a) Find the equation of the tangent plane to the surface at the point $(-2, 1, 1)$ to $\ln(x^2 + y^2 + z^2 + 1) + xyz = 3$.

b) Find the equation of the normal line to the surface at this point.

c) What are the coordinates of the line when it intersects the yz plane? (Hint: $x = 0$ here.)

<u>Local Maxima and Minima</u> To find local maxima and minima we need to first locate all critical points. A critical point is the set of all points where

$$\begin{aligned} f_x &= 0 \\ f_y &= 0 \end{aligned}$$

(x_0, y_0) is a local maximum if $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 > 0$ and $f_{xx}(x_0, y_0) < 0$

(x_0, y_0) is a local minimum if $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 > 0$ and $f_{xx}(x_0, y_0) > 0$

(x_0, y_0) is a saddle if $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 < 0$.
Let $D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$

Examples:

7. Find and label all local extrema if

$$\begin{aligned} f(x, y) &= xy^3 + 12y^2 - 8x \\ f_x &= y^3 - 8 = 0 \Rightarrow y = 2 \\ f_y &= 3xy^2 + 24y = 0 \; 3x(2)^2 + 24(2) = 0 \Rightarrow x = -4 \end{aligned}$$

The only critical point is $(-4, 2)$. Now

$$\begin{aligned} f_{xx} &= 0 \\ f_{xy} &= 3y^2 \\ f_{yy} &= 6xy + 24 \end{aligned}$$

At $(-4, 2)$

$$\begin{aligned} f_{xx} &= 0 \\ f_{xy} &= 12 \\ f_{yy} &= -24 \end{aligned}$$

and $D = 0(-24) - (12)^2 < 0 \Rightarrow (-4, 2)$ is a saddle point.

8.

$$\begin{aligned} f(x, y) &= 2y^3 + x^2y + 5y^2 + x^2 \\ f_x &= 2xy + 2x = 0 \Rightarrow 2x(y + 1) = 0 \text{ Either } x = 0 \text{ or } y = -1 \\ f_y &= 6y^2 + x^2 + 10y = 0 \end{aligned}$$

Putting these values into f_y we see when $x = 0 \Rightarrow 6y^2 + 10y = 0 \Rightarrow y = 0, -\dfrac{10}{6}$ when $y = -1 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2$. There are four critical points $(0, 0)$, $\left(0, -\dfrac{5}{3}\right)$, $(2, -1)$, $(-2, -1)$.

Now

$$f_{xx} = 2y + 2$$
$$f_{yy} = 12y + 10$$
$$f_{xy} = 2x$$

$D = (2y+2)(12y+10) - (2x)^2$

at (0,0), $D = 20 - 0 > 0 \qquad f_{xx} = 2 > 0 \Rightarrow (0,0)$ is a minimum.

at $\left(0, -\frac{5}{3}\right), D = \frac{40}{3} - 0 > 0 \qquad f_{xx} = -\frac{4}{3} < 0 \Rightarrow \left(0, -\frac{5}{3}\right)$ is a maximum.

at $(2,-1)$, $D = 0 - (4)^2 < 0 \Rightarrow (2,-1)$ is a saddle.

at $(-2,-1)$, $D = 0 - (4)^2 < 0 \Rightarrow (-2,-1)$ is a saddle.

Problems:

19. Find all critical points of the following function and use the second derivative test to determine which are relative maxima, relative minima, and saddle points.

$$f(x,y) = 25xy - x^3y - xy^2$$

20. Find all critical points of the following function and use the second derivative test to determine which are relative maxima, relative minima, and saddle points.

$$f(x,y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$$

21. Find all critical points of the following function and use the second derivative test to determine which are relative maxima, relative minima, and saddle points.

$$f(x,y) = x^3 - 3xy^2 - 3x^2 - 3y^2 + 1$$

22. Find all critical points of the following function and use the second derivative test to determine which are relative maxima, relative minima, and saddle points.

$$f(x,y) = xye^{-x^2-y^2}$$

23. Find all critical points of the following function and use the second derivative test to determine which are relative maxima, relative minima, and saddle points.

$$f(x,y) = \frac{x^2y^2}{2} - \frac{y^3}{3} - x^4 - 10x^2 + y^2 + 5$$

24. Find all critical points of the following function and use the second derivative test to determine which are relative maxima, relative minima, and saddle points.

$$f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$$

25. Find all critical points of the following function and use the second derivative test to determine which are relative maxima, relative minima, and saddle points.

$$f(x,y) = 3xy - x^2y - y^2$$

26. Find all critical points of the following function and use the second derivative test to determine which are relative maxima, relative minima, and saddle points.

$$f(x,y) = \frac{x^3}{3} + xy^2 + 5x^2 + y^2$$

Absolute Extrema This occurs when $f(x,y)$ is on a bounded domain. The absolute maximum and minimum occur either at a critical point or on the boundary.

Examples:

9. $f(x,y) = xy^2$ on $D = \{(x,y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 3\}$

First find critical points

$$f_x = y^2 = 0$$
$$f_y = 2xy = 0$$

Critical points are $(x, 0)$. Now check f on the boundary.

$$f(0,y) = 0$$
$$f(x,0) = 0$$

on $x^2 + y^2 = 3$ $\qquad f = x(3 - x^2)$.

Then $f' = 3 - 3x^2 = 0 \Rightarrow x = \pm 1$ $\qquad x = -1$ doesn't work (outside region).
So $x = +1 \Rightarrow y = \pm\sqrt{2}$ but $y = -\sqrt{2}$ doesn't work (outside region).

Need to evaluate f at these points.

$$f(0,0) = 0$$
$$f(0,\sqrt{3}) = 0$$
$$f(\sqrt{3},0) = 0$$
$$f(1,\sqrt{2}) = 1(\sqrt{2})^2 = 2$$

10. $f = 4x + 6y - x^2 - y^2$ on $D = \{(x,y) : 0 \leq x \leq 4, 0 \leq y \leq 5\}$

$$\begin{aligned} f_x &= 4 - 2x = 0 \Rightarrow x = 2 \\ f + y &= 6 - 2y = 0 \Rightarrow y = 3 \end{aligned} \Rightarrow \text{critical point is at } (2,3)$$

The region looks like

Evaluate f on the boundary.

Along $x = 0$ with $0 \leq y \leq 5$, $f(0, y) = 6y - y^2$ then $f' = 6 - 2y = 0 \Rightarrow y = 3$ and we have 3 points (0,0), (0,5) and (0,3).

Along $y = 5$ with $0 \leq x \leq 4$, $f(x, 5) = 4x - x^2 - 5$ then $f' = 4 - 2x = 0 \Rightarrow x = 2$ and we have three points (0,5), (4,5) and (2,5).

Along $x = 4$ with $0 \leq y \leq 5$, $f(4, y) = 6y - y^2$ then $f' = 6 - 2y = 0 \Rightarrow y = 3$ and we have three points (4,0), (4,5) and (4,3).

Finally along $y = 0$ with $0 \leq x \leq 4$, $f(x, 0) = 4x - x^2$ then $f' = 4 - 2x = 0 \Rightarrow x = 2$ and we have three points (0,0), (4,0) and (2,0).

We need to evaluate f at these points. Let *cp* denote critical points and *ep* denote end points.

	cp	$(2,3)$	$f(2,3) = 13$
$x = 0$	*ep*	$(0,0)$	$f(0,0) = 0$
	ep	$(0,5)$	$f(0,5) = 5$
	cp	$(0,3)$	$f(0,3) = 9$
$y = 5$	*ep*	$(0,5)$	done
	ep	$(4,5)$	$f(4,5) = 5$
	cp	$(2,5)$	$f(2,5) = 9$
$x = 4$	*ep*	$(4,0)$	$f(4,0) = 0$
	ep	$(4,5)$	done
	cp	$(4,3)$	$f(4,3) = 9$
$y = 0$	*ep*	$(0,0)$	done
	ep	$(4,0)$	done
	cp	$(2,0)$	$f(2,0) = 4$

The largest value is 13 and the smallest value is 0.

Problems:

27. Find the absolute extrema of the function $f(x, y) = 25xy - x^3y - xy^2$ on the region bounded by $0 \leq y \leq 4$ and $0 \leq x \leq 3$.

28. Find the absolute extrema of the function $f(x, y) = 2y^3 + 3y^2 - 12y + x^3 - 12x + 1$ on the region bounded by $y = 0,\ x = 3$, and $y = x$.

29. Find the absolute extrema of the function $f(x, y) = -x^3 + 4xy - 2y^2 + 1$ on the region bounded by $x^2 \leq y \leq 3$.

30. Find the absolute extrema of the function $f(x, y) = x^4 - y^2 - \frac{1}{16}x + \frac{1}{16}y$ on the region bounded by $x^2 \leq y \leq x$.

31. Find the absolute extrema of the function $f(x, y) = x^2 - 4xy + 2y^2 + 4x$ on the region bounded by $0 \leq y \leq 2x$ and $0 \leq x \leq 3$.

32. Find the absolute extrema of the function $f(x, y) = 5 + 4x - 2x^2 + 3y - y^2$ on the region bounded by $y = 2,\ y = x$ and $y = -x.$.

33. Find the absolute extrema of the function $f(x, y) = 3xy - x^2y - y^2$ on the region bounded by $y = -1,\ x = 1$ and $y = 1 + x$. Hint: See problem 4.

34. Find the absolute extrema of the function $f(x, y) = \dfrac{x^2}{2} + xy$ on the region bounded by $y = 4 - x^2$ and $y = 3$.

<u>Lagrange Multipliers</u> Minimize or maximize a function subject to a constraint.

Example: Maximize $V = xyz$ subject to $2xy + 2y + xz = 24$.

$$\begin{aligned} F &= xyz - \lambda(2xy + 2yz + xz - 24) \\ F_x &= yz - 2\lambda y - \lambda z = 0 \\ F_y &= xz - 2\lambda x - 2\lambda z = 0 \\ F_z &= xy - 2\lambda x - \lambda x = 0 \\ F_\lambda &= -(2xy + 2yz + xz - 24) = 0 \end{aligned}$$

Need to solve these simultaneous equations

$$xF_x - yF_y = -\lambda xz + 2\lambda yz = 0 \Rightarrow z(\lambda)(-x + 2y) = 0$$

If $\lambda = 0 \Rightarrow x = y = z = 0$. If $z = 0 \Rightarrow x = y = 0$ so $x = 2y$.

$$xF_x - zF_z = -2\lambda xy + 2\lambda xz = 0 \Rightarrow 2\lambda x(-y + z) = 0$$

Again $\lambda \neq 0,\ x \neq 0$ so $z = y$. Putting these into the F_λ equation we see that

$$\begin{aligned} 2(2y)y + 2y(y) + (2y)(y) - 24 = 0 \\ 4y^2 + 2y^2 + 2y^2 = 24 \\ 8y^2 = 24\ y\ \sqrt{3}\ z = \sqrt{3},\ x = 2\sqrt{3} \\ \text{and } V = 2(3)^{3/2} \end{aligned}$$

Problems:

35. Find the minimum value of $f(x, y) = 4xy$ subject to the constraint $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$.

36. Find the minimum value of $f(x, y) = e^{xy}$ subject to the constraint $x^2 + y^2 = 8$.

37. Find the minimum value of $f(x, y) = x^2 - 8xy + y^2 - 12y + 48$ subject to the constraint $x + y = 8$.

38. Find the minimum value of $f(x, y) = \dfrac{1}{x} + \dfrac{1}{y}$ subject to the constraint $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 1$.

39. Find the minimum value of $f(x, y) = 2x^2 - 6xy + 3y^2 + 2x + 5$ subject to the constraint $2x - y = 3$.

40. Find the minimum value of $f(x, y) = xyz$ subject to the constraint $x + y + z^2 = 16$.
Hint: $F = F(x, y, z, \lambda)$.

41. Find the minimum value of $f(x, y, z) = x^2 + 2y^2 + 3z^2$ subject to the constraint $4x + 6y - 12z = 82$.

42. Find the minimum value of $f(x, y, z) = xy + 2xz + 2yz$ subject to the constraint $xyz = 64$.

43. Find the shortest distance from the point $(2, -2, 3)$ to the plane $6x + 4y - 3z = 2$.

i.e. Minimize $f(x, y, z) = (x - 2)^2 + (y + 2)^2 + (z - 3)^2$ subject to $6x + 4y - 3z = 2$.

This will give you the point. Then find the distance between these two points.

44. In class we considered Lagrange Multipliers with one constraint. Suppose we have 2 constraints. Consider the problem:

$$\begin{array}{cc} \text{Maximize} & f(x,y,z) = xyz \\ \text{subject to} & x^2 + z^2 = 5 \text{ and } x - 2y = 0. \end{array}$$

Then $F(x, y, z, \lambda, \mu) = xyz - \lambda(x^2 + z^2 - 5) - \mu(x - 2y)$.

a) Compute $F_x, F_y, F_z, F_\lambda, F_\mu$

b) Now find the critical points by setting these equal to 0 and solve for x, y, z, λ, μ.