Math 250 **Polynomial Phun**

Consider a polynomial of order n

$$p_0 r^n + p_1 r^{n-1} + \cdots + p_{n-1} r + p_n = 0$$

where p_i are real constants.
Facts:

1. If n is odd there is at least one real root (may be hard to find!).

2. Start at the lefthand side of the equation and count the number of sign changes between coefficients. Suppose this number is m. Then there are m or $m-2$ or $m-4$ or $\cdots$ positive roots.

3. Replace x by $(-x)$ and count the number of sign changes again. Let this number be s. Then there are s or $s-2$ or $s-4$ or $\cdots$ negative roots.

4. Determine the factors of p_0 and p_n. If there are rational roots then they must be in the form of $\pm\dfrac{q_n}{q_0}$ where q_n is a factor of p_n and q_0 is a factor of q_n.

Example A. $x^5 + 3x^3 + x + 1 = 0$

1. At least one root.

2. No sign changes $\Rightarrow$ no positive roots.

3. $(-x)^5 + 3(-x)^3 + (-x) + 1 = -x^5 - 3x^3 - x + 1 = 0$
   One sign change $\Rightarrow$ one root and it's negative.

4. Root is not rational. Since $q_0 = 1,\ q_n = 1$ and -1 is not a root.

Example B. $x^6 + 4x^4 + x^2 + 1 = 0$

1. No sign changes $\Rightarrow$ no positive roots.

2. $(-x)^6 + 4(-x)^4 + (-x)^2 + 1 = x^6 + 4x^4 + x^2 + 1 = 0$
   No sign change $\Rightarrow$ no negative roots.

3. No real roots.

Example C. $x^3 - 2x^2 - x + 2 = 0$

1. At least one root.

2. 2 sign changes $\Rightarrow$ 2 or 0 positive roots.

3. $(-x^3) - 2(-x)^2 - (-x) + 2 = -x^3 - 2x^3 + x + 2 = 0$
   1 sign change $\Rightarrow$ one negative root.

$$q_n = 1, 2 \qquad q_0 = 1$$

If there are rational roots then they must be of the form $\pm\frac{2}{1}, \pm\frac{1}{1}$.

Try

$x = -2 \quad (-2)^3 - 2(-2)^2 - (-2) + 2 = -8 - 8 + 2 + 2 \neq 0$

$x = 2 \quad 2^3 - 2(2)^3 - 2 + 2 = 0 \Rightarrow x = 2$ is a root. (Notice there must be a second positive root.)

$x = 1 \quad 1^3 - 2(1)^2 - 1 + 2 = 0 \Rightarrow x = 1$ is a root.

$x = -1 \quad (-1)^3 - 2(-1)^2 - (-1) + 2 = 0 \Rightarrow x = -1$ is a root.

Problems:

1. $2x^3 - 5x^2 + 7x - 2$

2. $x^3 - 12x^2 + 48x - 64 = 0$

3. $x^{10} + x^8 + 1 = 0$

Finding all roots of a real number

Example D. Suppose you want to solve $r^3 + 8 = 0$. You want to find the 3 roots of -8.

$$-8 = 8[\cos(\pi + 2n\pi) + i\sin(\pi + 2n\pi)] = 8e^{i(\pi+2n\pi)}$$

$$(-8)^{1/3} = \left[8e^{i(\pi+2n\pi)}\right]^{1/3} = 8^{1/3}e^{i\left(\dfrac{\pi+2n\pi}{3}\right)}$$

$$= 2\left[\cos\frac{\pi+2n\pi}{3} + i\sin\frac{\pi+2n\pi}{3}\right]$$

$$\begin{aligned}
n =& 0 \qquad 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = 2\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right] \\
n =& 1 \qquad 2[\cos\pi + i\sin\pi] = 2[-1+0i] \\
n =& 2 \qquad 2\left[\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right] = 2\left[\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right]
\end{aligned}$$

Example E. Suppose you want to solve $r^4 - 4 = 0$. You want to find the 4 roots of 4.

$$4 = 4[\cos(2n\pi) + i\sin(2n\pi)] = 4e^{i2n\pi}$$

$$(4)^{1/4} = [4e^{i2n\pi}]^{1/4} = 4^{1/4}e^{\dfrac{i2n\pi}{4}}$$

$$= \sqrt{2}\left[\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right]$$

$$\begin{aligned}
n &= 0 \qquad \sqrt{2}[\ \ 1 + i0] \\
n &= 1 \qquad \sqrt{2}[\ \ 0 + i1] \\
n &= 2 \qquad \sqrt{2}[-1 + i0] \\
n &= 3 \qquad \sqrt{2}[\ \ 0 + i(-1)]
\end{aligned}$$

Problems:

4. $r^4 + 2 = 0$

5. $r^6 - 27 = 0$

6. $r^8 + 256 = 0$