

K.A. Pericak-Spector

There are 3 line integral formulas.

I. $\int_C f(x, y, z) ds$ (Evaluation of a line integral as a definite integral.)

II. $\int_C (f(x, y, z) dx + g(x, y, z) dy + h(x, y, z) dz)$ (Evaluating a line integral in differential form.)

III. $\int_C \mathbf{F} \cdot d\mathbf{r}$ (Evaluating a vector-valued function over a curve C .)

Each involve integrating over a curve (NOT a region). Sometimes you will have to determine the parameterization of C . Sometimes not.

Case I. Here you need C written as a parameterized curve. Put f in terms of t and $ds = \sqrt{(x')^2 + (y')^2 + (z')^2} dt$.

Example. $\int_c x ds$ $C : x = 1 - t, y = (1 - t)^2$ $0 \leq t \leq 1$

Step 1. Write the integrand in terms of t : $x = (1 - t)$

Step 2. Compute $\sqrt{(x')^2 + (y')^2} = \sqrt{(-1)^2 + (2(1 - t))^2}$

Step 3. Evaluate $\int_0^1 x(t) \sqrt{(x')^2 + (y')^2} dt = \int_0^1 (1 - t) \sqrt{1 + 4(1 - t)^2} dt$

A. $\int_C (x + 2) ds$ $C : x = t$ $y = \frac{4}{3} t^{3/2}$ $z = \frac{t^2}{2}$ $0 \leq t \leq 2$

Step 1. Write the integral in terms of t .

Step 2. Compute $\sqrt{(x')^2 + (y')^2 + (z')^2}$

Step 3. Evaluate integral.

B. $\int_C (x^2 + y^2) ds$ C : line from $(0,0)$ to $(3,0)$.

Step 1. Draw C .

Step 2. Parameterize C .

Step 3. Write integrand in terms of t .

Step 4. Compute $\sqrt{(x')^2 + (y')^2}$

Step 5. Evaluate integral.

Case II. Now you want to evaluate a line integral in differential form.

$$\text{Rewrite } \int_C \left[f(x(t), y(t), z(t)) \frac{dx}{dt} + g(x(t), y(t), z(t)) \frac{dy}{dt} + h(x(t), y(t), z(t)) \frac{dz}{dt} \right] dt$$

Example. $\int_C (xydx + y dy)$ $C : x = 4t \quad y = t \quad 0 \leq t \leq 1$

Step 1. Compute $\frac{dx}{dt}, \frac{dy}{dt}$ here $\frac{dx}{dt} = 4, \frac{dy}{dt} = 1$

Step 2. Write integrand in terms of t : $(xy \frac{dx}{dt} + y \frac{dy}{dt}) = (4t)(t)(4) + t(1)$

Step 3. Evaluate: $\int_0^1 (16t^2 + t) dt$

A. $\int_C xydx + ydy$ $C : x = 4 \cos t \quad y = 4 \sin t \quad 0 \leq t \leq \frac{\pi}{2}$

Step 1. Compute $\frac{dx}{dt}, \frac{dy}{dt}$.

Step 2. Write integrand in terms of t .

Step 3. Evaluate integral.

B. $\int_C (x^2 y dx + (y - z) dy + xyz dz)$ $C : x = t \quad y = t^2 \quad z = 2 \quad 0 \leq t \leq 2$

Step 1. Compute $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$.

Step 2. Write integrand in terms of t .

Step 3. Evaluate integral.

C. $\int_C (-y dx - x dy)$ $C : x = 2 \cos t \quad y = 2 \sin t \quad 0 \leq t \leq \frac{\pi}{2}$

Step 1. Compute $\frac{dx}{dt}, \frac{dy}{dt}$.

Step 2. Write integrand in terms of t .

Step 3. Evaluate integral.

Case III. Here \mathbf{F} and $d\mathbf{r}$ are vectors.

Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ if

A. $\mathbf{F} = 3x\mathbf{i} + 4y\mathbf{j}$ $\mathbf{r} = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j}$ $0 \leq t \leq \frac{\pi}{2}$

Step 1. $x = 2 \cos t$ $y = 2 \sin t$ (x and y are the components of \mathbf{r}). Write \mathbf{F} in terms of t .

Step 2. Compute \mathbf{r}' .

Step 3. Compute $\mathbf{F} \cdot \mathbf{r}'$.

Step 4. Evaluate integral $\int_0^{\pi/2} \mathbf{F} \cdot \mathbf{r}' dt$.

B. $\mathbf{F} = 3y \mathbf{i} + 4x \mathbf{j}$ $\mathbf{r} = \sqrt{4 - t^2} \mathbf{i} + t \mathbf{j}$ $-2 \leq t \leq 2$

Step 1. $x = t$ $y = \sqrt{4 - t^2}$. Write \mathbf{F} in terms of t .

Step 2. Compute \mathbf{r}' .

Step 3. Compute $\mathbf{F} \cdot \mathbf{r}'$.

Step 4. Evaluate integral $\int_2^{-2} \mathbf{F} \cdot \mathbf{r}' dt$.

C. $\mathbf{F} = x^2 \mathbf{i} + y^2 \mathbf{j} + z^2 \mathbf{k}$ $\mathbf{r} = (\sin t) \mathbf{i} + (\cos t) \mathbf{j} + t^2 \mathbf{k}$ $0 \leq t \leq \frac{\pi}{2}$

Step 1. $x = \sin t$ $y = \cos t$ $z = t^2$. Write \mathbf{F} in terms of t .

Step 2. Compute \mathbf{r}' .

Step 3. Compute $\mathbf{F} \cdot \mathbf{r}'$.

Step 4. Evaluate integral $\int_0^{\pi/2} \mathbf{F} \cdot \mathbf{r}' dt$.

The following vector-valued functions are conservative. Find f such that $\mathbf{F} = \nabla f$.

$$\mathbf{F} = \frac{1}{2}xy \mathbf{i} + \frac{1}{4}x^2 \mathbf{j}$$

$$\mathbf{F} = 2xy \mathbf{i} + (x^2 - y) \mathbf{j}$$

$$\mathbf{F} = e^x(\sin y \mathbf{i} + \cos y \mathbf{j})$$

$$\mathbf{F} = 2xy \mathbf{i} + (x^2 + z^2) \mathbf{j} + 2zy \mathbf{k}$$

$$\mathbf{F} = e^x \cos y \mathbf{i} - e^x \sin y \mathbf{j} + 2 \mathbf{k}$$