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Math 251 Worksheet for 13.8 Name________________________

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Stoke's Theorem: Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_S\int \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$$

Basically Stoke's Theorem involves
1) Surface integral or closed curve integral.
2) Surface is open or curve is closed.
3) In surface integral the integrand involves a vector-valued function; vector-valued function can be written as the curl of a vector.

Example 1. Compute the surface integral $\int_S\int \operatorname{curl}\mathbf{w}\cdot\mathbf{N}ds$ where S is the paraboloid $z = 16 - x^2 - y^2,\ x^2+y^2 \le 16$, the z component of $\mathbf{N}$ points in the $\langle 0,0,1\rangle$ direction, and $\mathbf{w}$ is the vector field

$$\mathbf{w} = \langle \sec y, x\tan y\sec y + x,\ e^{xz}\sec y\cos z\rangle$$

Step 1. Is the surface open? Here, yes.

Step 2. Does the integrand involve the curl of a vector? Here, yes.

Step 3. Identify the curve C. Here $C:\ x^2+y^2 = 16$.

Step 4. Identify $\mathbf{r}$. Here $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + 0\mathbf{k}$

Step 5. Take $d\mathbf{r}$. Here $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + 0\mathbf{k}$

Step 6. Identify $\mathbf{F}$. Here $\mathbf{F} = \langle \sec y, x\tan y\sec y + x, \sec y\rangle$. (From above $z = 0$.)

Step 7. Compute $\mathbf{F}\cdot d\mathbf{r}$. Here $\mathbf{F}\cdot d\mathbf{r} = \sec y dx + (x\tan y\sec y + x)dy$.

Step 8. Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$. Here $\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (\sec y dx + (x\tan y\sec y + x)dy)$.

IN THIS CASE we can use Green's Theorem.

$$P = \sec y,\ Q = x\tan y \sec y + x,\ D : x^2 + y^2 \leq 16$$
$$\oint_C (\sec y dx + (x\tan y \sec y + x)dy) = \int_D\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$
$$= \int_D\int dA = \pi(4)^2 = 16\pi$$

Example 2: Compute the surface integral $\int_S\int \mathbf{v}\cdot\mathbf{N}d\mathbf{S}$ where S is the parabold $z = 16 - x^2 - y^2$, $x^2 + y^2 \leq 9$, the z component of $\mathbf{N}$ points in the $\langle 0,0,1\rangle)$ direction and $\mathbf{v}$ is the vector field.

$$\mathbf{v} = \text{curl}\langle z\sec y,\ 7x\sec y\tan y + z^2x,\ xyz\rangle$$

Step 1. Is the surface open? Here, yes.

Step 2. Does the integrand involve the curl of a vector? Here, yes.

Step 3. Identify the curve C. Here $C : x^2 + y^2 = 9$.

Step 4. Identify $\mathbf{r}$. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + 7\mathbf{k}$ $(z = 16 - (x^2 + y^2) = 16 - 9 = 7.)$

Step 5. Take $d\mathbf{r}$. Here $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{i} + 0\mathbf{k}$

Step 6. Identify $\mathbf{F}$. Here $\mathbf{F} = \langle 7\sec y,\ 7x\sec y\tan y + 49x,\ 7xy\rangle$.

Step 7. Compute $\mathbf{F}\cdot d\mathbf{r}$. Here $\mathbf{F}\cdot d\mathbf{r} = 7\sec y dx + (7x\sec y\tan y + 49x)dy$.

Step 8. Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$. Here $\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (7\sec y ds + (7x\sec y\tan y + 49x)dy)$

In this case we can once again use Green's Theoreom.

$$\oint)_C\mathbf{F}\cdot d\mathbf{r} = \oint_C (7\sec y ds + (7x\sec y\tan y + 49x)dy)$$
$$= \int_R\int 49dA \text{ where} R : x^2 + y^2 \leq 9$$
$$= 49\pi(3)^2$$

Problem 1. Compute the surface integral $\int_S\int \operatorname{curl} \mathbf{w} \cdot \mathbf{N} dS$ where S is the paraboloid $z = 4 - x^2 - y^2$, $z \geq 0$, the z component of $\mathbf{N}$ points in the $\langle 0, 0, 1 \rangle$ direction, and $\mathbf{w}$ is the vector field.

$$\mathbf{w} = 2z\mathbf{i} + x\mathbf{j} + y^2\mathbf{k}$$

Step 1. Is the surface open?

Step 2. Does the integrand involve the curl of a vector?

Step 3. Identify the curve C.

Step 4. Identify $\mathbf{r}$.

Step 5. Take $d\mathbf{r}$.

Step 6. Identify $\mathbf{F}$.

Step 7. Compute $\mathbf{F} \cdot d\mathbf{r}$.

Step 8. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Problem 2. Compute the surface integral $\int_S\int \operatorname{curl} \mathbf{w} \cdot \mathbf{N} dS$ where S is the paraboloid $z = 4 - x^2 - y^2$, $z \geq 1$, the z component of $\mathbf{N}$ points in the $\langle 0, 0, 1 \rangle$ direction, and $\mathbf{w}$ is the vector field.

$$\mathbf{w} = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$$

Step 1. Is the surface open?

Step 2. Does the integrand involve the curl of a vector?

Step 3. Identify the curve C. Here $C:\ x^2 + y^2 = 16$.

Step 4. Identify $\mathbf{r}$.

Step 5. Take $d\mathbf{r}$.

Step 6. Identify $\mathbf{F}$.

Step 7. Compute $\mathbf{F} \cdot d\mathbf{r}$.

Step 8. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

The following examples will not result in using Green's Theorem in part 8. You will need to parameterize curve.

Problem 3. Compute the surface integral $\int_S \int \operatorname{curl} \mathbf{w} \cdot \mathbf{N} dS$ where S is the plane in the first octant $3x + 4y = 2z = 12$, the z component of $\mathbf{N}$ points in the $\langle 0, 0, 1 \rangle$ direction, and $\mathbf{w}$ is the vector field.

$$\mathbf{w} = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Step 1. Is the surface open?

Step 2. Does the integrand involve the curl of a vector?

Step 3. Identify the curve C. (Here there will be 3 curves.)

Step 4. Identify $\mathbf{r}$. (Here there will be 3 $\mathbf{r}$'s.)

Step 5. Take $d\mathbf{r}$.

Step 6. Identify $\mathbf{F}$.

Step 7. Compute $\mathbf{F} \cdot d\mathbf{r}$.

Step 8. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$. Here $\oint \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r})$.

Problem 4. Compute the surface integral $\int_S \int \operatorname{curl} \mathbf{w} \cdot \mathbf{N} dS$ where S is the plane in the first octant $s : z = x^2 \; 0 \leq x \leq a \; 0 \leq y \leq a$, the z component of $\mathbf{N}$ points in the $\langle 0, 0, 1 \rangle$ direction, and $\mathbf{w}$ is the vector field. (This region is defined by 4 curves.)

$$\mathbf{w} = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$$

Step 1. Is the surface open?

Step 2. Does the integrand involve the curl of a vector?

Step 3. Identify the curve C. (Here there will be 4 curves.)

Step 4. Identify $\mathbf{r}$. (Here there will be 4 $\mathbf{r}$'s.)

Step 5. Take $d\mathbf{r}$.

Step 6. Identify $\mathbf{F}$.

Step 7. Compute $\mathbf{F} \cdot d\mathbf{r}$.

Step 8. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$. Here $\oint \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r}$.